Tree Pólya Splitting distributions for multivariate count data

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Introduction

The analysis of multivariate count data is fundamental in various fields.

- Ecology: Joint Species Distribution Models;
- Genomics: RNA sequences;
- Insurance: Number of claims.

Introduction

An appropriate model should be:

- Flexible for various correlations;
- Simple for inference and interpretation;
- Taking into account overdispersion.

Introduction

Discrete Schur-constant [Castañer et al. (2015)]

Let $\mathbf{Y} := (Y_1, \ldots, Y_J) \in \mathbb{N}^J$ be a discrete random vector. It has a Schur-constant joint survival function if, for all $\mathbf{y} = (y_1, \ldots, y_J) \in \mathbb{N}^J$,

$$\mathbb{P}(\mathbf{Y} \geq \mathbf{y}) = S(|\mathbf{y}|),$$

where $S : \mathbb{N} \mapsto [0, 1]$ is a valid function, and $|\mathbf{y}| := y_1 + \cdots + y_J$.

Introduction

Under this hypothesis, Castañer et al. [2015] prove that $\mathbf{Y}$ is Schur-constant if there exists a univariate discrete distribution $\mathcal{L}(\psi)$ such that

$$|\mathbf{Y}| \sim \mathcal{L}(\psi) \text{ and } \mathbf{Y} \mid |\mathbf{Y}| = n \sim \mathcal{DM}_{\Delta_n}(\mathbf{1}),$$

where $\mathcal{DM}_{\Delta_n}(\mathbf{1})$ denotes the Dirichlet-multinomial distribution with parameters $\mathbf{1} = (1, \ldots, 1)$ and support $\Delta_n := \{\mathbf{y} \in \mathbb{N}^J : |\mathbf{y}| = n\}$, the discrete simplex.

Introduction

Birth–death process under extended neutral theory [Peyhardi et al. (2024)]

As an application to joint species distribution models, Peyhardi et al. [2024] studied the stationary distribution of a multivariate birth-death process under the ecological neutrality assumption.

Introduction

In this framework, they proved that the random vector $\mathbf{Y}$ at the stationary state satisfies

$$|\mathbf{Y}| \sim \mathcal{L}(\psi) \quad \text{and} \quad \mathbf{Y} \mid |\mathbf{Y}| = n \sim \mathcal{P}_{\Delta_n}^{[c]}(\boldsymbol{\theta}),$$

where $\mathcal{P}_{\Delta_n}^{[c]}(\boldsymbol{\theta})$ denotes the Pólya distribution with parameters $c \in \{-1, 0, 1\}$ and $\boldsymbol{\theta} \in \mathbb{R}_+^J$.

Introduction

These special cases can all be represented by the following construction proposed by Jones & Marchand [2019] and Peyhardi et al. [2021]:

1. The sum $|\mathbf{Y}| := Y_1 + \cdots + Y_J$ follows a univariate discrete distribution $\mathcal{L}(\psi)$;
2. Given $|\mathbf{Y}| = n$, $\mathbf{Y}$ follows a Pólya distribution on the discrete simplex Δ_n.

Introduction

They name this class of distributions the *Sums and Shares* or the *Pólya Splitting*. Various distributions are particular cases, such as:

- Poisson-multinomial distribution;
- Negative multinomial distribution;
- Multivariate generalized Waring distribution [Xekalaki (1986)];
- Counts of Bernoulli success strings [Ait Aoudia et al. (2016)].

Introduction

This class of distributions, however, is quite restrictive since **all pairwise correlations must have identical signs**.

In this work, we sought to address this limitation by generalizing the Pólya Splitting model.

Goals:

1. Define and build the *Tree Pólya Splitting* model;
2. Study its properties, particularly its dependence structure;
3. Apply the model to the Trichoptera dataset.

Contents

Pólya Splitting model

Pólya Splitting model

Let $c \in \{-1, 0, 1\}$, the generalized factorial is given by

$$(x)_{(n,c)} = \begin{cases} 1 & \text{if } n = 0, \\ x(x+c)\dots(x+(n-1)c) & \text{if } n \geq 1. \end{cases}$$

As special cases, we have the following:

- Falling factorial: $(x)_{(n)} := (x)_{(n,-1)}$;
- Rising factorial: $(x)_n := (x)_{(n,1)}$.

Pólya Splitting model

Let $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_J) \in \mathbb{R}_+^J$, the probability mass function (p.m.f.) of the *Pólya* distribution $\mathcal{P}_{\Delta_n}^{[c]}(\boldsymbol{\theta})$ is given by

$$\mathbb{P}(\mathbf{Y} = \mathbf{y}) = \frac{n!}{(|\boldsymbol{\theta}|)_{(n,c)}} \prod_{j=1}^{J} \frac{(\theta_j)_{(y_j,c)}}{y_j!} \mathbb{1}_{\Delta_n}(\mathbf{y}).$$

Pólya Splitting model

The three particular cases of $\mathcal{P}_{\Delta_n}^{[c]}(\boldsymbol{\theta})$ are given by

- $c = -1$: Multivariate hypergeometric distribution $\mathcal{H}_{\Delta_n}(\boldsymbol{\theta})$;
- $c = 0$: Multinomial distribution $\mathcal{M}_{\Delta_n}(\boldsymbol{\theta})$;
- $c = 1$: Dirichlet-multinomial distribution $\mathcal{DM}_{\Delta_n}(\boldsymbol{\theta})$.

Pólya Splitting model

Similarly, for the parameters θ, γ and $y \in \{0, \ldots, n\}$, the *univariate Pólya* has p.m.f.

$$\mathbb{P}(Y = y) = \frac{n!}{(\theta + \gamma)_{(n,c)}} \frac{(\theta)_{(y,c)}}{y!} \frac{(\gamma)_{(n-y,c)}}{(n-y)!} \mathbb{1}_{\{0,\ldots,n\}}(y).$$

We denote the latter by $\mathcal{P}_n^{[c]}(\theta, \gamma)$, and its particular cases are:

- $c = -1$: Hypergeometric distribution $\mathcal{H}_n(\theta, \gamma)$;
- $c = 0$: Binomial distribution $\mathcal{B}_n(\theta, \gamma)$;
- $c = 1$: Beta-binomial distribution $\mathcal{BB}_n(\theta, \gamma)$.

Pólya Splitting model

The random vector $\mathbf{Y} = (Y_1, \ldots, Y_J) \in \mathbb{N}^J$ has a *Pólya Splitting* distribution if

$$\mathbf{Y} \sim \mathcal{P}^{[c]}_{\Delta_n}(\boldsymbol{\theta}) \underset{n}{\wedge} \mathcal{L}(\boldsymbol{\psi}),$$

where $\wedge$ is the mixture operator. Its p.m.f. is given by

$$\mathbb{P}(\mathbf{Y} = \mathbf{y}) = \mathbb{P}(|\mathbf{Y}| = n) \left[\frac{n!}{(|\boldsymbol{\theta}|)_{(n,c)}} \prod_{j=1}^{J} \frac{(\theta_j)_{(y_j,c)}}{y_j!} \right],$$

where $n = |\mathbf{y}|$ and $\mathbb{P}(|\mathbf{Y}| = n)$ is the p.m.f. of $\mathcal{L}(\boldsymbol{\psi})$.

Pólya Splitting model - Marginal

Theorem [Peyhardi et al. (2021)]

Let $\mathbf{Y} \sim \mathcal{P}^{[c]}_{\Delta_n}(\boldsymbol{\theta}) \underset{n}{\wedge} \mathcal{L}(\boldsymbol{\psi})$ and $\mathcal{I} \subseteq \{1, \dots, J\}$. The subvector $\mathbf{Y}_{\mathcal{I}}$ follows the distribution

$$\mathbf{Y}_{\mathcal{I}} \sim \mathcal{P}^{[c]}_{\Delta_n}(\boldsymbol{\theta}_{\mathcal{I}}) \underset{n}{\wedge} \left[\mathcal{P}^{[c]}_{m}(|\boldsymbol{\theta}_{\mathcal{I}}|, |\boldsymbol{\theta}_{-\mathcal{I}}|) \underset{m}{\wedge} \mathcal{L}(\boldsymbol{\psi}) \right].$$

In particular, the marginal distribution Y_j is given by

$$Y_j \sim \mathcal{P}^{[c]}_{n}(\theta_j, |\boldsymbol{\theta}_{-j}|) \underset{n}{\wedge} \mathcal{L}(\boldsymbol{\psi}).$$

Note: The univariate marginal is equivalent to the thinning operator in discrete time series (see Peyhardi [2023]).

Pólya Splitting model - Covariance & correlation

Theorem [Peyhardi et al. (2021)]

Let $\mathbf{Y} \sim \mathcal{P}^{[c]}_{\Delta_n}(\boldsymbol{\theta}) \underset{n}{\wedge} \mathcal{L}(\boldsymbol{\psi})$ and marginals Y_i, Y_j $(i \neq j)$. Then, the covariance is given by

$$\mathrm{Cov}(Y_i, Y_j) = \frac{\theta_i \theta_j}{|\boldsymbol{\theta}|^2 (|\boldsymbol{\theta}| + c)} \left[(\mu_2 - \mu_1^2) |\boldsymbol{\theta}| - c\mu_1^2 \right],$$

where μ_r the r-th factorial moment of $\mathcal{L}$, i.e.

$$\mu_r := \mathrm{E}\left[(|\mathbf{Y}|)_{(r)} \right].$$

Note: $\mu_2 - \mu_1^2 = \mathrm{Var}\left[|\mathbf{Y}|\right] - \mathrm{E}\left[|\mathbf{Y}|\right]$, which determines the dispersion of $\mathcal{L}$ and the sign of the covariance.

Pólya Splitting model - Covariance & correlation

Theorem [Valiquette et al. (2025)]

Let $\mathbf{Y} \sim \mathcal{P}^{[c]}_{\Delta_n}(\boldsymbol{\theta}) \underset{n}{\wedge} \mathcal{L}(\boldsymbol{\psi})$ and marginals Y_i, Y_j $(i \neq j)$. Then, the Pearson correlation is given by

$$\mathrm{Corr}(Y_i, Y_j) = \mathrm{sgn}(1 - M_i)\sqrt{\frac{\theta_i \theta_j}{(\theta_i + c)(\theta_j + c)}}(1 - M_i)(1 - M_j),$$

where

$$M_k = \frac{\mu_1 \left(1 + \frac{c}{|\boldsymbol{\theta}|}\mu_1\right)}{\mu_2 \left(\frac{\theta_k + c}{|\boldsymbol{\theta}| + c}\right) + \mu_1 \left(1 - \mu_1 \frac{\theta_k}{|\boldsymbol{\theta}|}\right)}, \quad k = i, j,$$

and $\mathrm{sgn}(x) = \mathbb{1}_{[0,\infty)} - \mathbb{1}_{(-\infty,0]}$.

Pólya Splitting model - Covariance & correlation

For $c = 1$, we can find an upper bound for this correlation, which generalizes the result of Jones & Marchand [2019].

Theorem [Valiquette et al. (2025)]

Let $\mathbf{Y} \sim \mathcal{DM}_{\Delta_n}(\boldsymbol{\theta}) \underset{n}{\wedge} \mathcal{L}(\boldsymbol{\psi})$ and $i \neq j$, then the Pearson correlation is such that

$$\mathrm{Corr}(Y_i, Y_j) < \sqrt{\frac{\theta_i \theta_j}{(\theta_i + 1)(\theta_j + 1)}}.$$

Tree Pólya Splitting model

Tree Pólya Splitting - Definitions

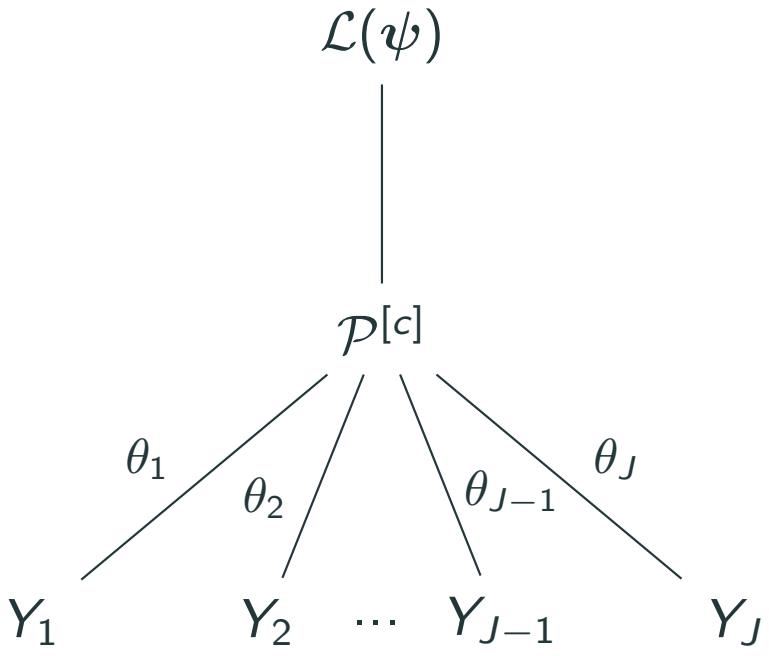
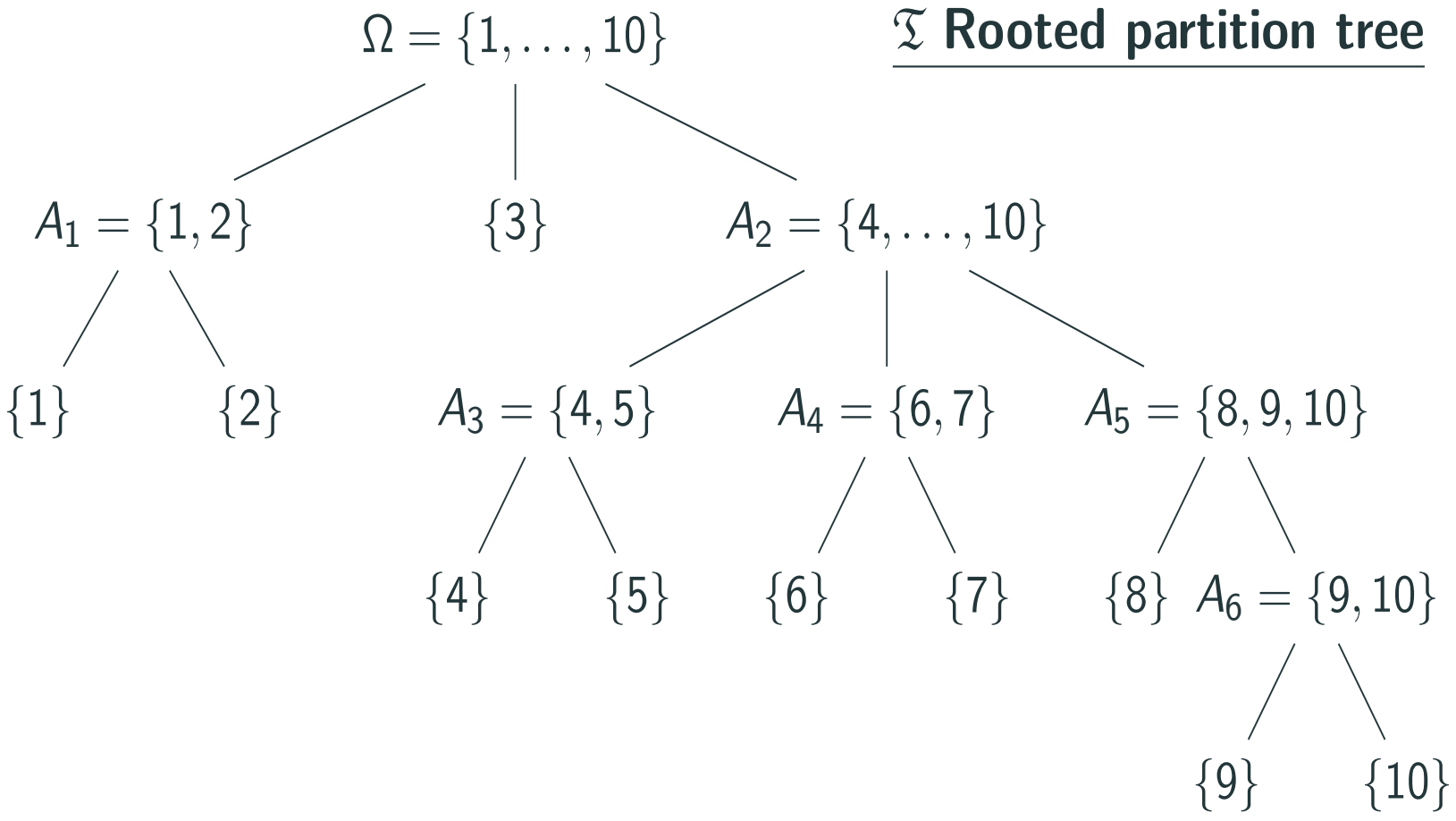
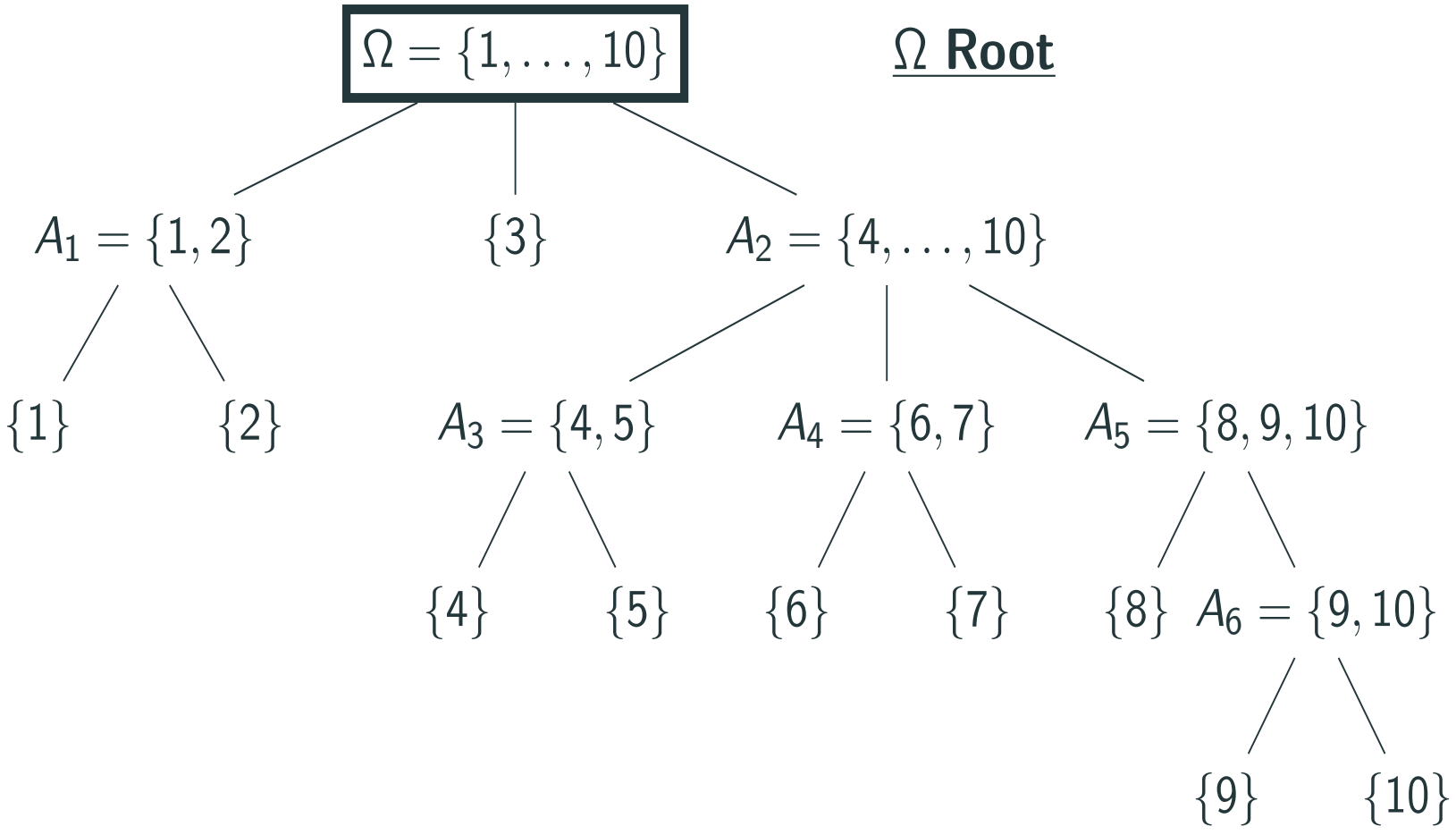


Figure 1: Representation of the Pólya Splitting model

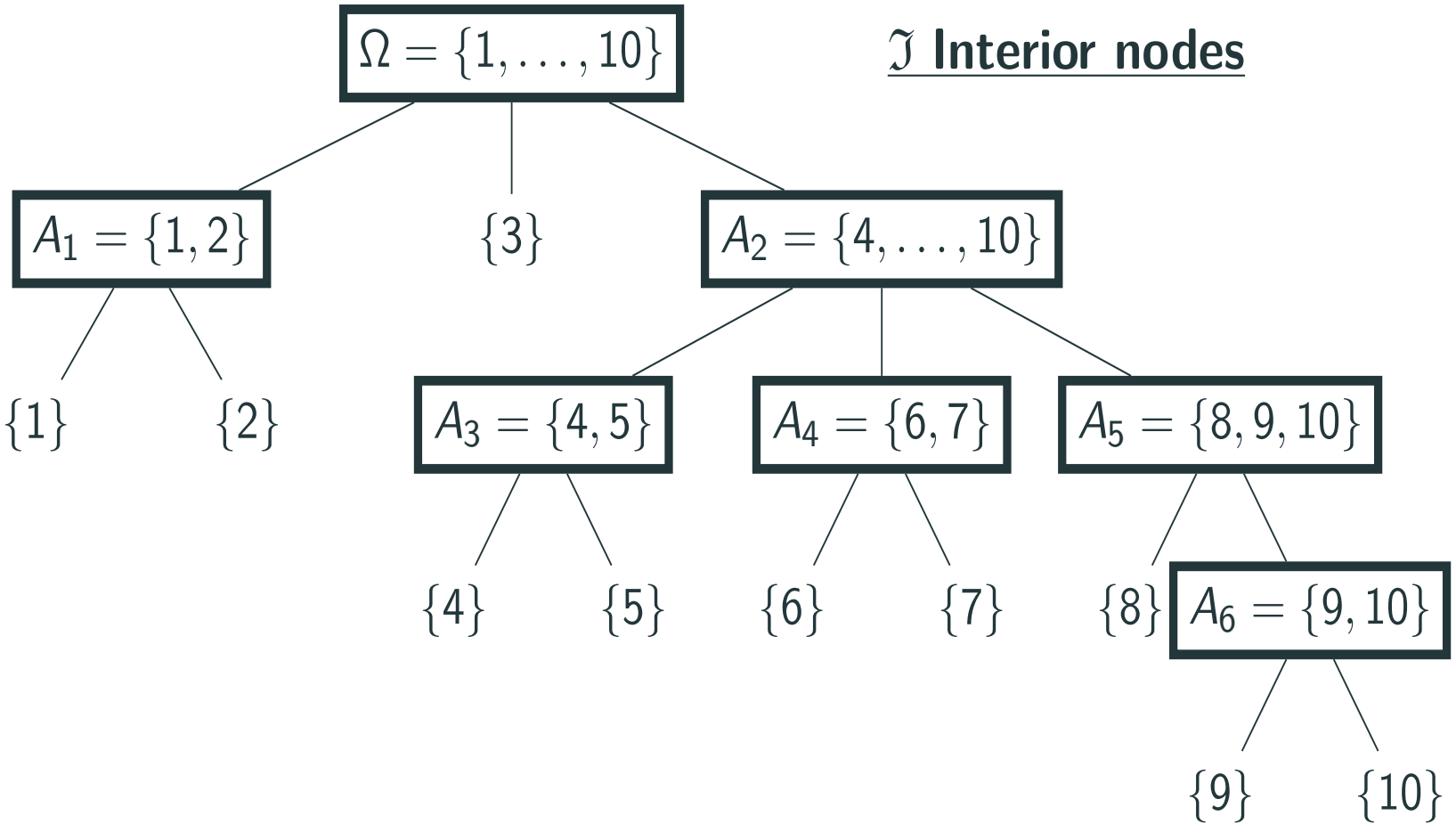
Tree Pólya Splitting - Definitions



Tree Pólya Splitting - Definitions



Tree Pólya Splitting - Definitions



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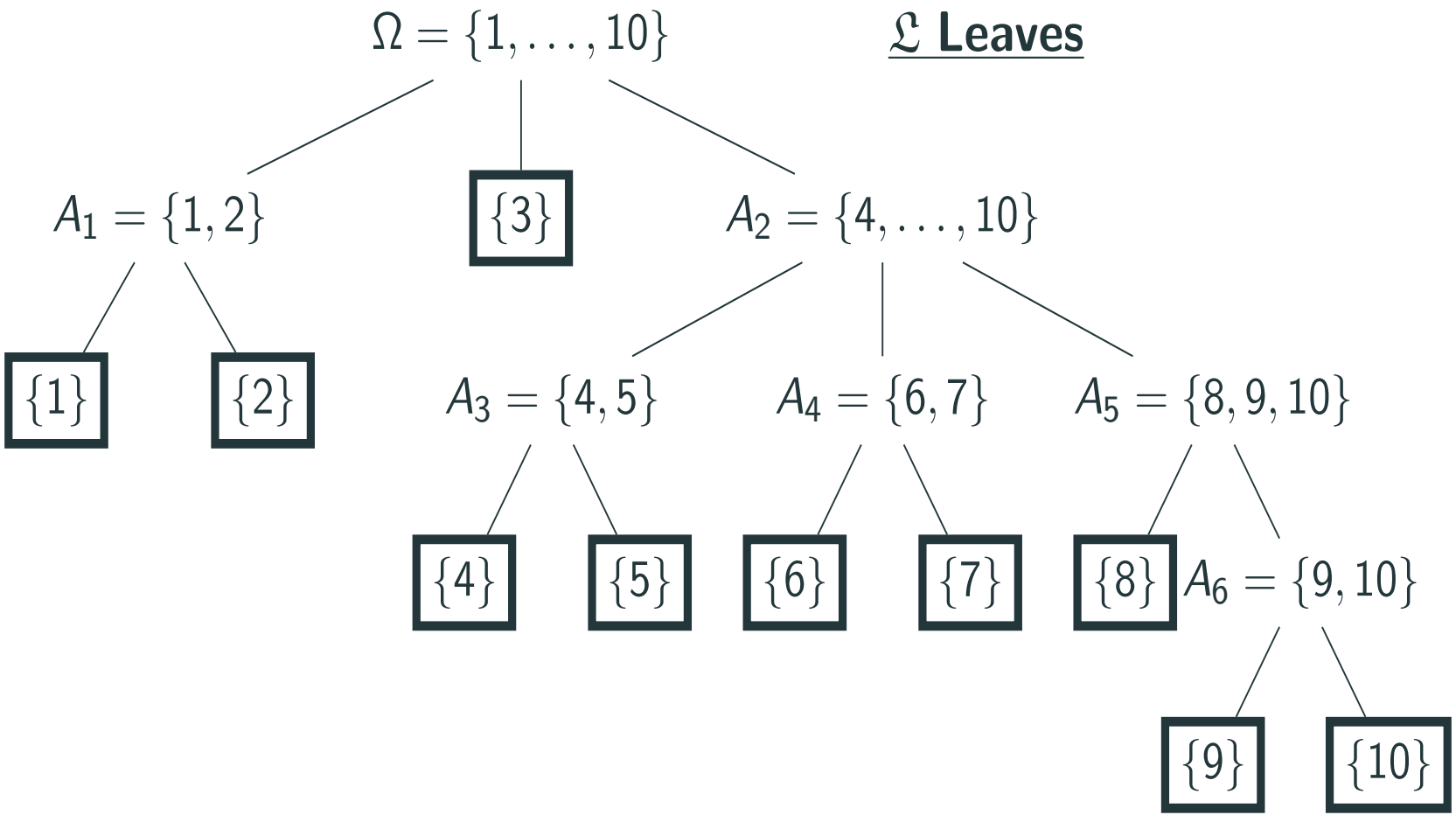

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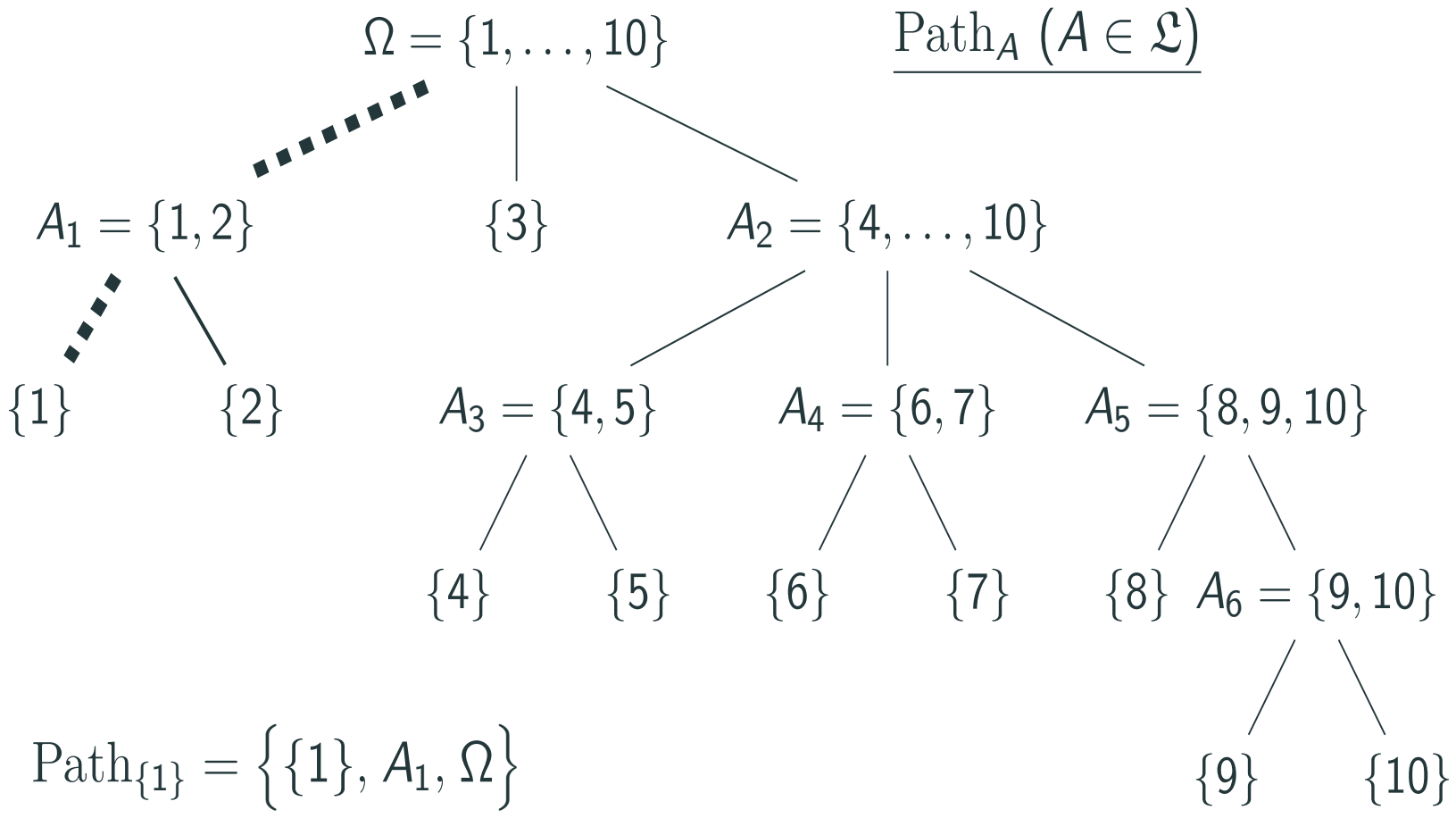
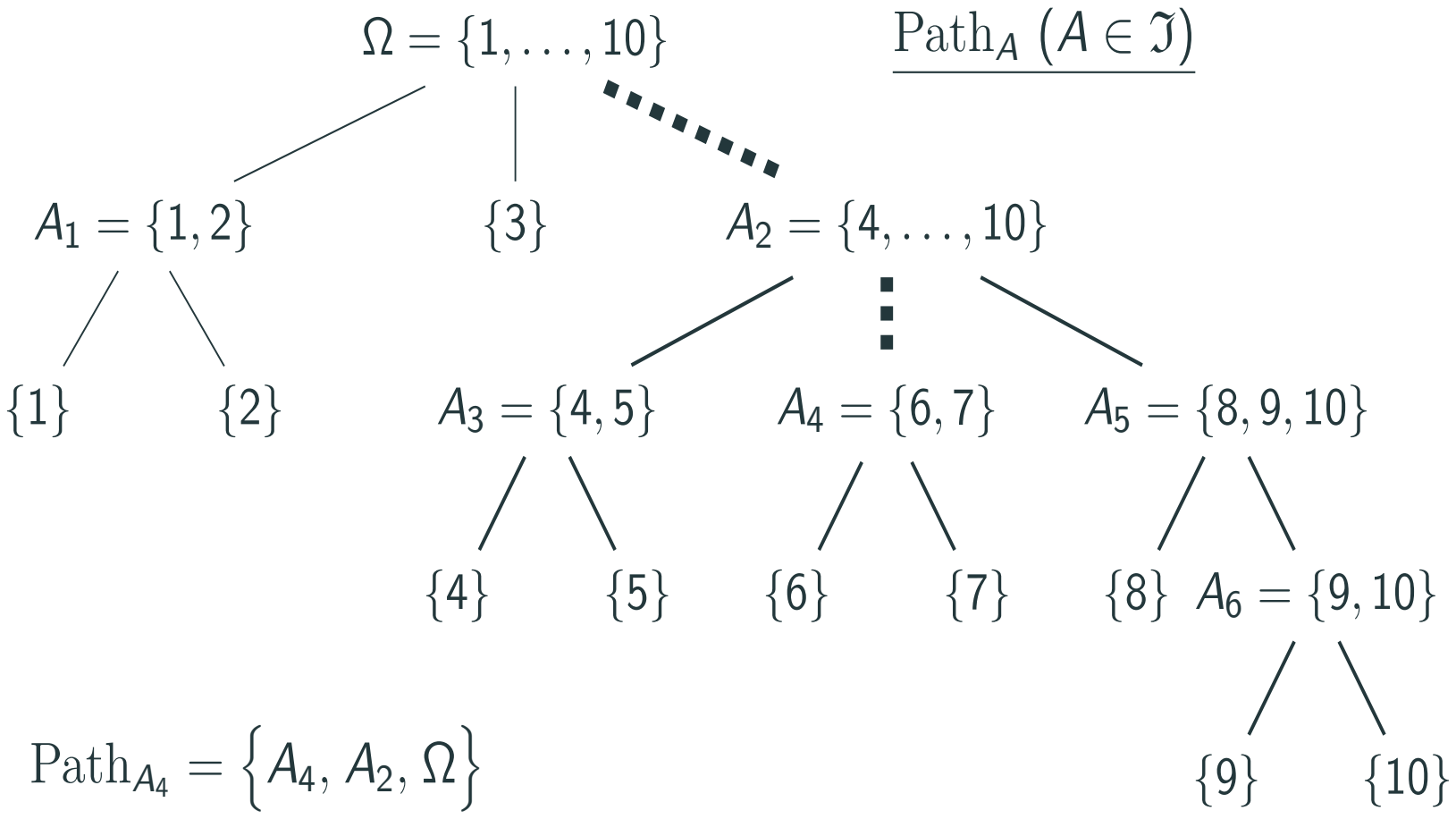


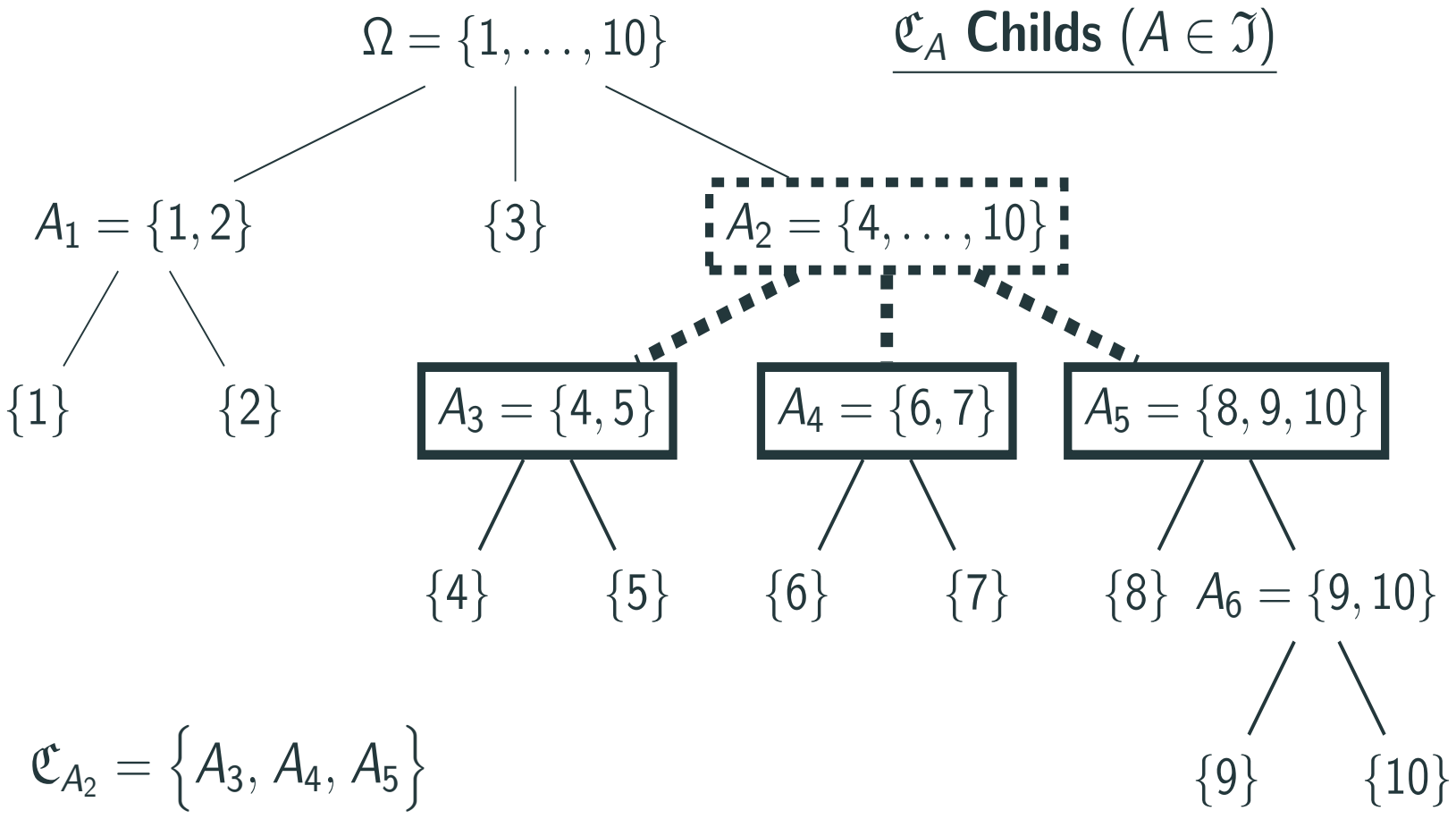
$$\mathrm{Path}_{\{1\}} = \Big\{\{1\}, A_1, \Omega\Big\}$$

Tree Pólya Splitting - Definitions



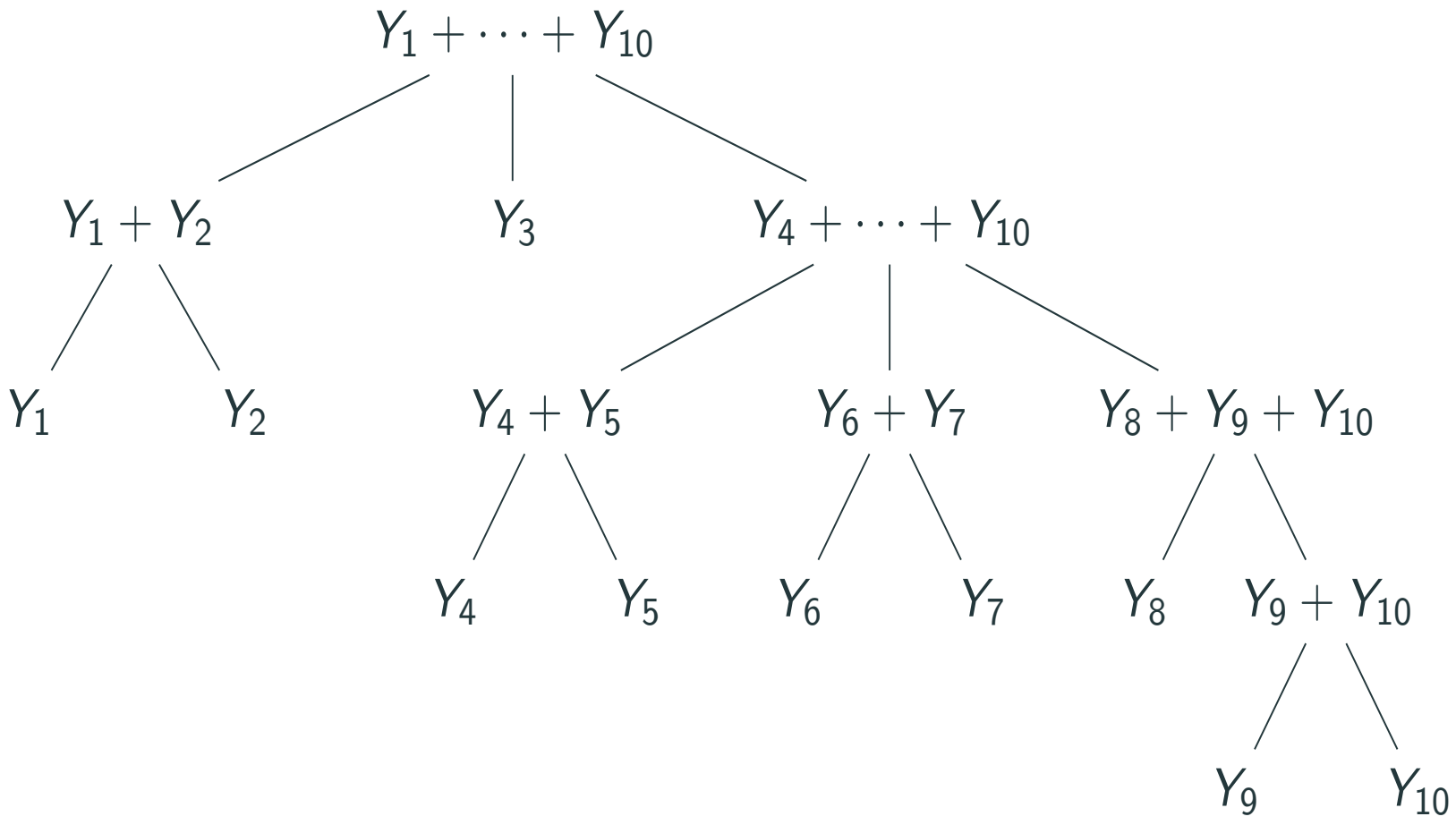
$$\mathrm{Path}_{A_4} = \left\{A_4, A_2, \Omega\right\}$$

Tree Pólya Splitting - Definitions

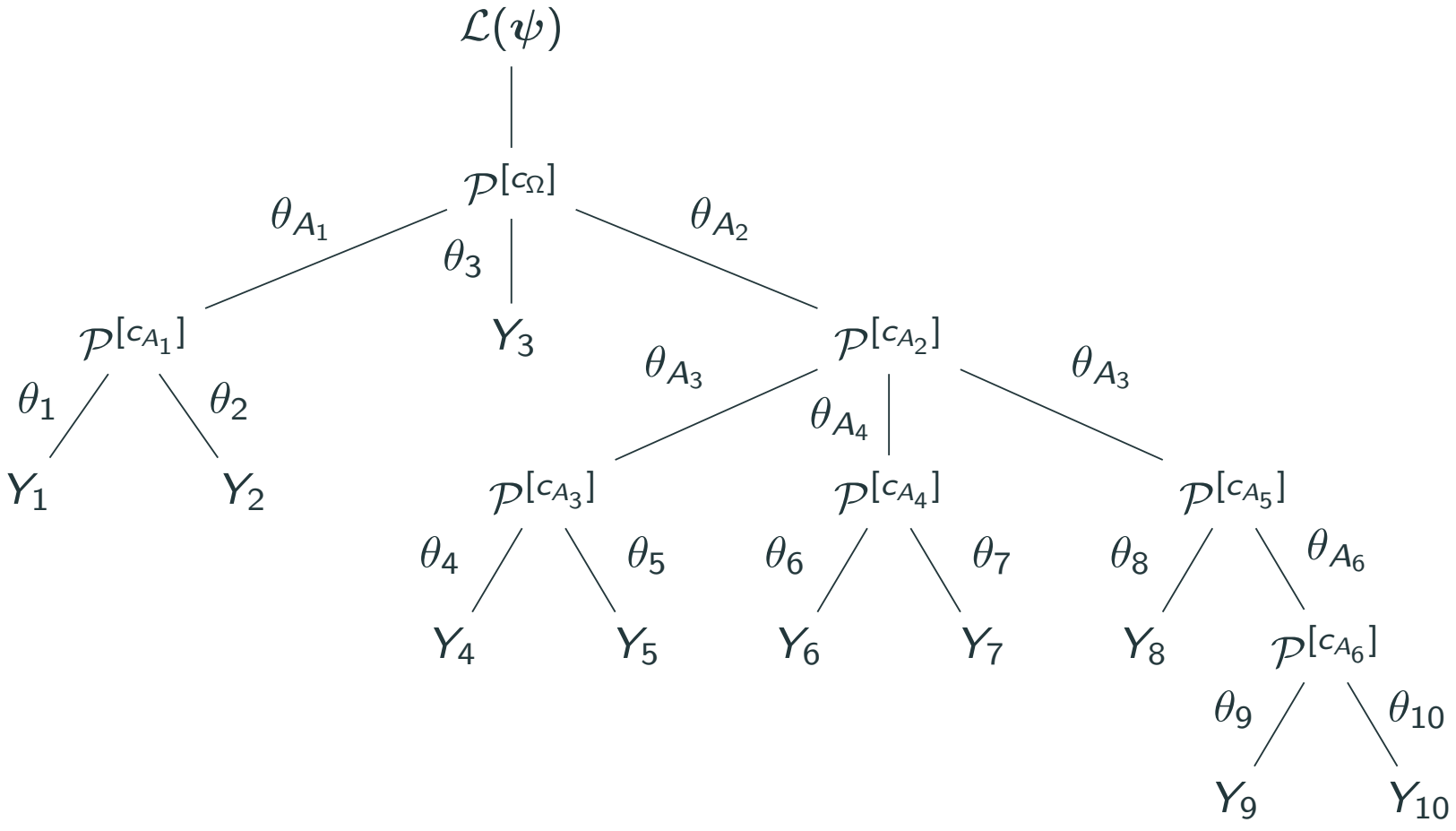


$$\mathfrak{C}_{A_2} = \{A_3, A_4, A_5\}$$

Tree Pólya Splitting - Definitions



Tree Pólya Splitting - Definitions



Tree Pólya Splitting - Definitions

Definition

The random vector $\mathbf{Y}$ has a *Tree Pólya Splitting* distribution with parameters $\mathfrak{T}$, $\boldsymbol{\theta} = (\boldsymbol{\theta}_A)_{A\in\mathfrak{I}}$, $\mathbf{c} = (c_A)_{A\in\mathfrak{I}}$, and ψ if its p.m.f. is given by

$$\mathbb{P}(\mathbf{Y}=\mathbf{y}) = \underbrace{\mathbb{P}(|\mathbf{Y}|=|\mathbf{y}|)}_{\mathcal{L}(\psi)} \prod_{A\in\mathfrak{I}} \underbrace{\frac{n_A!}{(|\boldsymbol{\theta}_A|)_{(n_A,c_A)}} \prod_{C\in\mathfrak{C}_A} \frac{(\theta_C)_{(n_C,c_A)}}{n_C!}}_{\mathcal{P}^{[c_A]}(\boldsymbol{\theta}_A)},$$

where $\mathbf{Y}_A = (Y_j)_{j\in A}$ and $n_A := |\mathbf{y}_A|$ for any node $A \in \mathfrak{I}$.

Tree Pólya Splitting - Definitions

Definition

Such a distribution is denoted by

$$\mathbf{Y} \sim \mathcal{TP}_{\Delta_n}(\mathfrak{T}; \boldsymbol{\theta}, \mathbf{c}) \underset{n}{\wedge} \mathcal{L}(\boldsymbol{\psi}).$$

As an example, let us suppose $\mathcal{L} = \mathcal{NB}(\alpha, p)$ with $\alpha > 0$, $p \in (0, 1)$, and p.m.f.

$$\mathbb{P}(|\mathbf{Y}| = n) = \frac{(\alpha)_n}{n!} p^n (1-p)^\alpha.$$

Moreover, let us use only multinomial ($\mathcal{M}$) or Dirichlet-multinomial ($\mathcal{DM}$) distributions in $\mathfrak{T}$.

Tree Pólya Splitting - Definitions

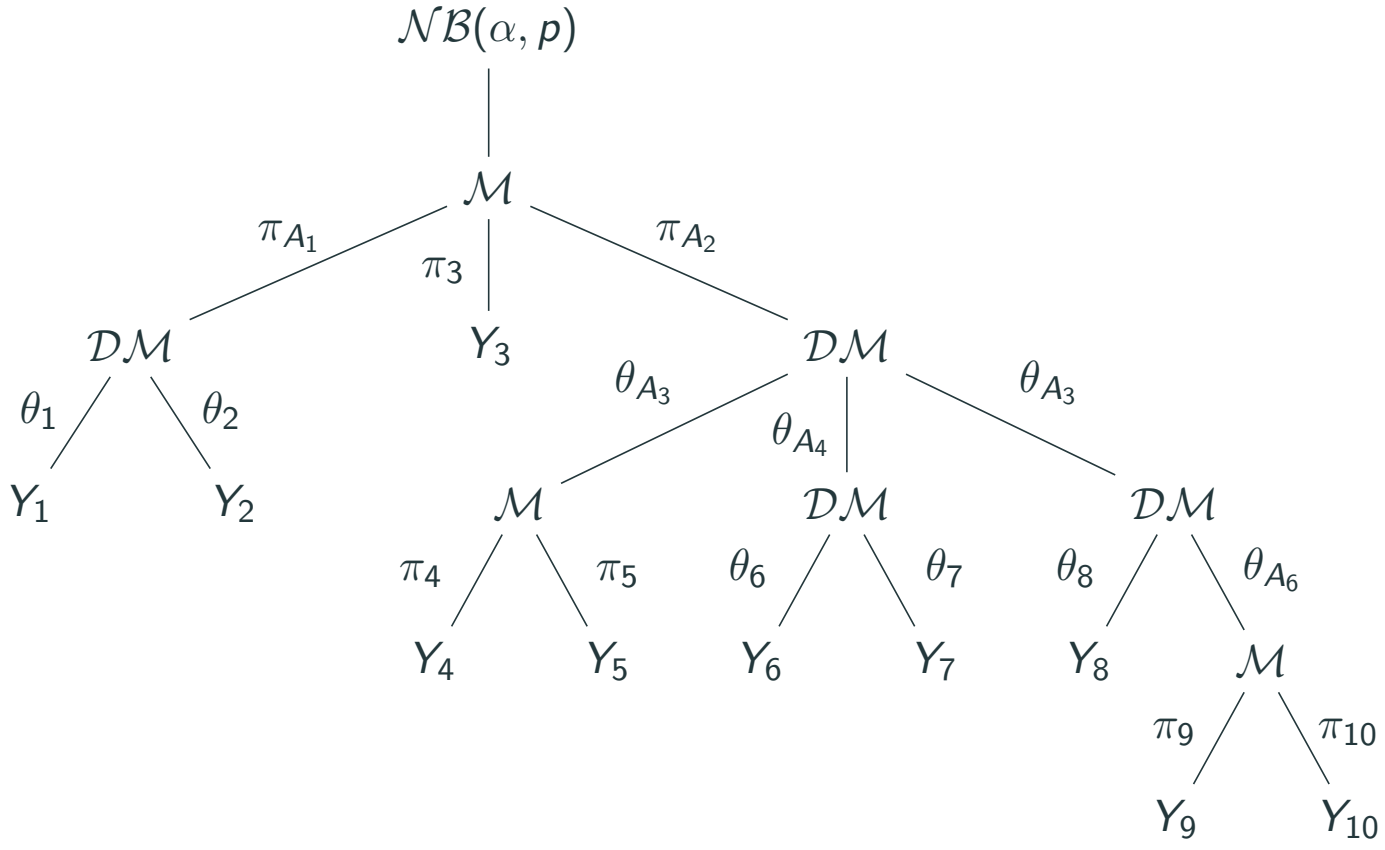


Figure 2: Exemple of a Tree Pólya Splitting model

Tree Pólya Splitting - Marginal

Theorem [Valiquette et al. (2025)]

Let $\mathbf{Y} \sim \mathcal{TP}_{\Delta_n}(\mathfrak{T}; \boldsymbol{\theta}, \mathbf{c}) \underset{n}{\wedge} \mathcal{L}(\boldsymbol{\psi})$. The marginal distribution of Y_j is given by

$$Y_j \sim \bigwedge_{k=1}^{K} \mathcal{P}_{n_k}^{[c_{A_k}]} \left(\theta_{A_{k-1}}, \left| \boldsymbol{\theta}_{A_k \setminus A_{k-1}} \right| \right) \underset{n_K}{\wedge} \mathcal{L}(\boldsymbol{\psi}), \tag{1}$$

where $A_k \in \mathrm{Path}_{\{j\}}$ and $\boldsymbol{\theta}_{A_k \setminus A_{k-1}}$ is the vector of parameters associated with the node A_k, such that $\theta_{A_{k-1}}$ is removed.

Tree Pólya Splitting - Marginal

Theorem [Valiquette et al. (2025)]

Let $\mathbf{Y} \sim \mathcal{TP}_{\Delta_n}(\mathfrak{T}; \boldsymbol{\theta}, \mathbf{c}) \underset{n}{\wedge} \mathcal{L}(\boldsymbol{\psi})$ and $A \in \mathfrak{I}$. The marginal distribution of $|\mathbf{Y}_A|$ is given by (1), where $A_k \in \mathrm{Path}_A$.

If $A = \Omega$, then $K = 0$ and we retrieve $|\mathbf{Y}| \sim \mathcal{L}(\boldsymbol{\psi})$.

Tree Pólya Splitting - Marginal

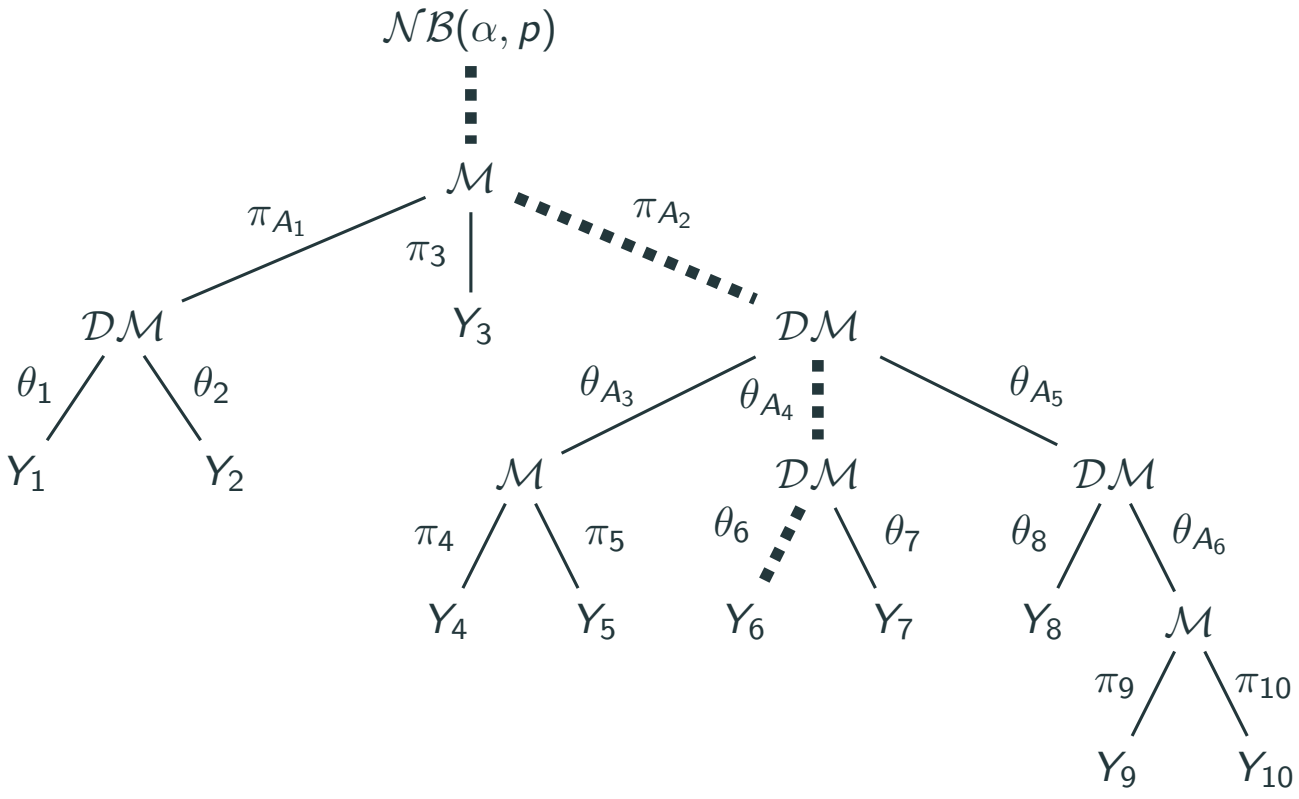
Theorem [Valiquette et al. (2025)]

Let $\mathbf{Y} \sim \mathcal{TP}_{\Delta_n}(\mathfrak{T}; \boldsymbol{\theta}, \mathbf{c}) \underset{n}{\wedge} \mathcal{L}(\psi)$ and $A \in \mathfrak{I}$. The multivariate marginal distribution of $\mathbf{Y}_A$ is given by

$$\mathbf{Y}_A \sim \mathcal{TP}_{\Delta_n}(\tilde{\mathfrak{T}}; \tilde{\boldsymbol{\theta}}, \tilde{\mathbf{c}}) \underset{n}{\wedge} |\mathbf{Y}_A|\,,$$

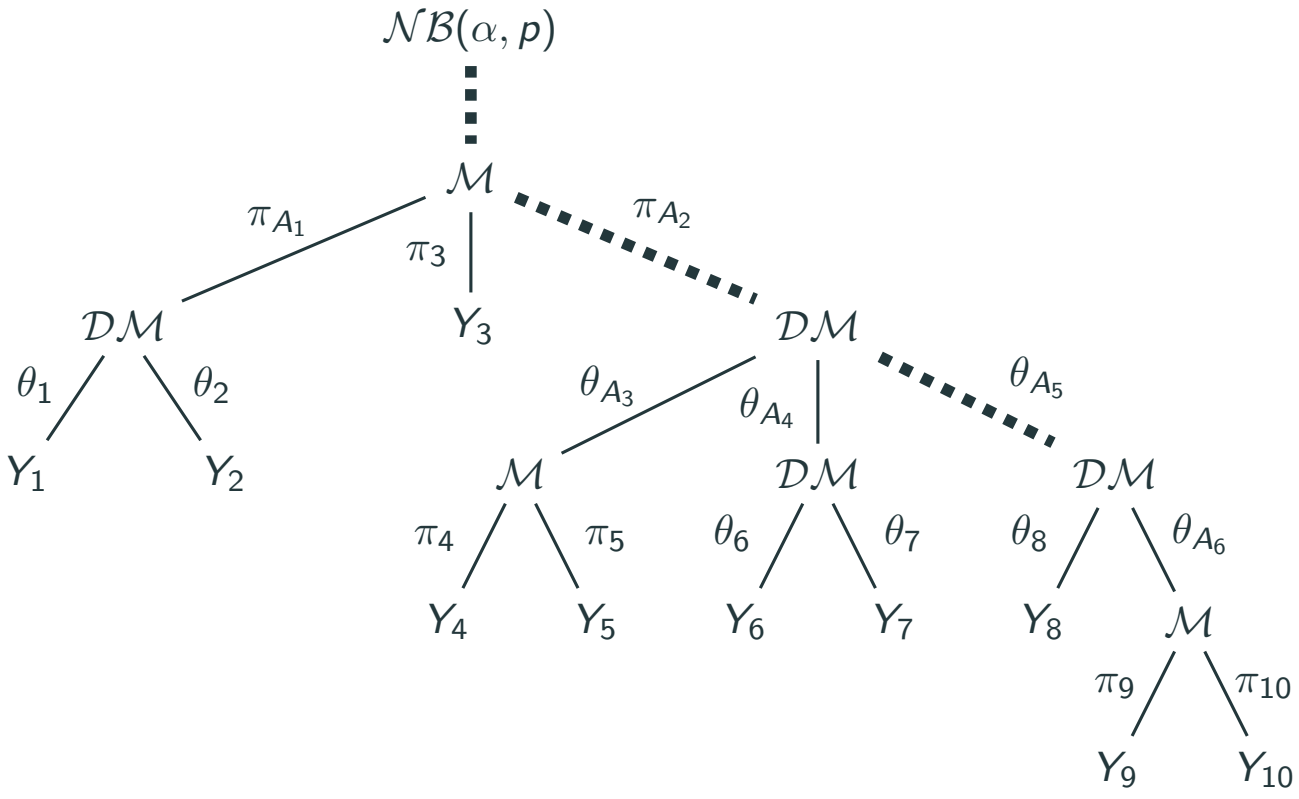
where $\tilde{\mathfrak{T}}$ is the pruned tree with root A, interior nodes $\tilde{\mathfrak{I}}$, leaves $\tilde{\mathfrak{L}}$, and parameters $\tilde{\boldsymbol{\theta}} = \{\boldsymbol{\theta}_A\}_{A \in \tilde{\mathfrak{I}}}$, $\tilde{\mathbf{c}} = \{\mathbf{c}_A\}_{A \in \tilde{\mathfrak{I}}}$.

Tree Pólya Splitting - Marginal



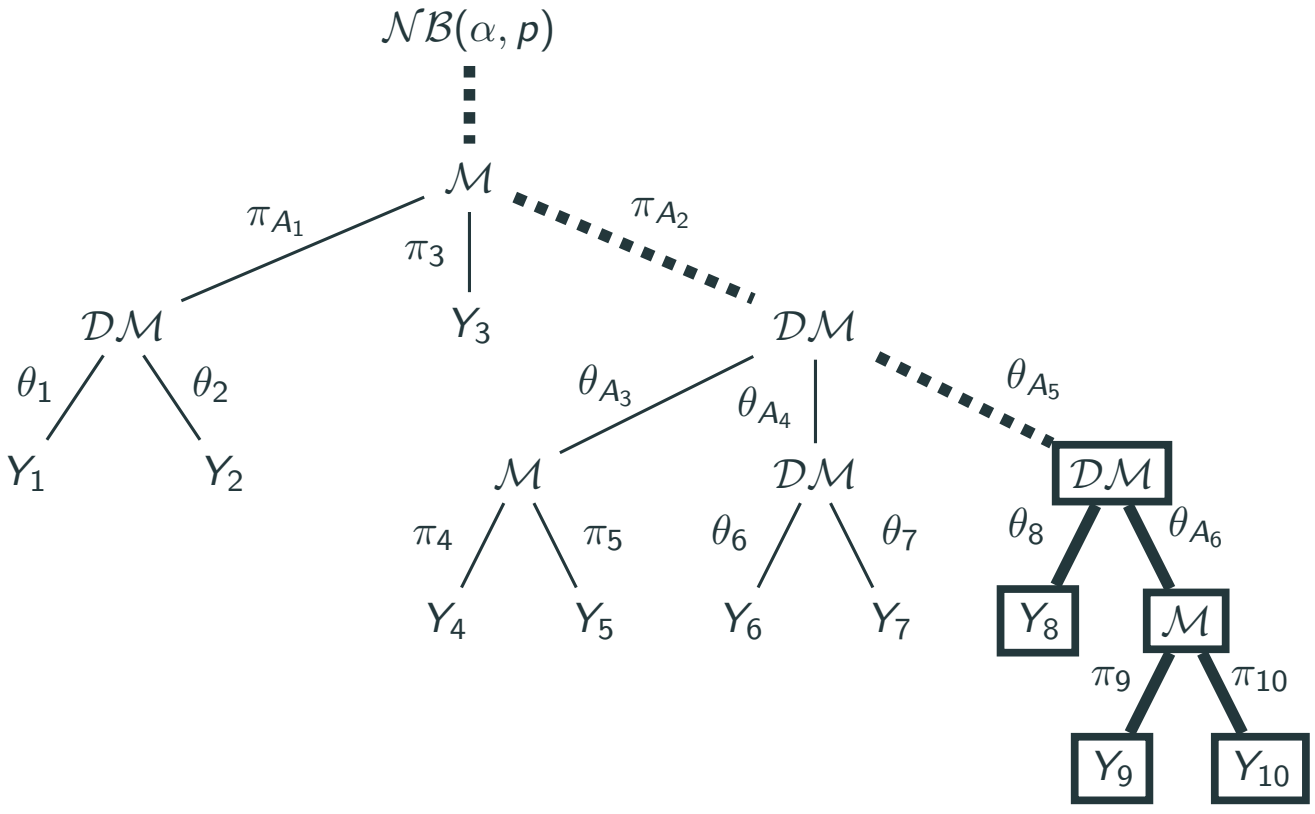
$$Y_6 \sim \mathcal{BB}_{n_1}(\theta_6, \theta_7) \underset{n_1}{\wedge} \mathcal{BB}_{n_2}(\theta_{A_4}, \theta_{A_3} + \theta_{A_5}) \underset{n_2}{\wedge} \mathcal{B}_{n_3}(\pi_{A_2}) \underset{n_3}{\wedge} \mathcal{NB}(\alpha, p)$$

Tree Pólya Splitting - Marginal



$$Y_8 + Y_9 + Y_{10} \sim \mathcal{BB}_{n_1}(\theta_{A_5}, \theta_{A_3} + \theta_{A_4}) \underset{n_1}{\wedge} \mathcal{B}_{n_2}(\pi_{A_2}) \underset{n_2}{\wedge} \mathcal{NB}(\alpha, p)$$

Tree Pólya Splitting - Marginal



$$(Y_8, Y_9, Y_{10}) \sim \mathcal{TP}_{\Delta_n}(\tilde{\mathfrak{T}}; \tilde{\boldsymbol{\theta}}, \tilde{\mathbf{c}}) \underset{n}{\wedge} (Y_8 + Y_9 + Y_{10})$$

Tree Pólya Splitting - Marginal

In this example, we can prove that all the distributions $\mathcal{B}_n$ can be "ignored". Therefore, when it comes to analyzing the marginal here, all of them have the form

$$X \sim \left[\bigwedge_{k=1}^{K} \mathcal{BB}_{n_k}(\alpha_k, \beta_k) \right] \underset{n_K}{\wedge} \mathcal{NB}(\alpha, p). \tag{2}$$

Tree Pólya Splitting - Marginal

For $K = 1$, Jones & Marchand [2019] prove that the p.m.f. of (2) is given by

$$\mathbb{P}(X = n) = (1-p)^r \frac{(r)_n (\alpha_1)_n}{(\alpha_1+\beta_1)_n} \frac{p^n}{n!} {}_2F_1\left[\begin{matrix} r+n,\ \beta_1 \\ \alpha_1+\beta_1+n \end{matrix}; p\right];\ n \in \mathbb{N},$$

where ${}_pF_q\left[\begin{matrix}\mathbf{a}\\ \mathbf{b}\end{matrix}; z\right]$ is the *generalized hypergeometric function* with $\mathbf{a} \in \mathbb{R}^p$, $\mathbf{b} \in \mathbb{R}_+^q$.

Tree Pólya Splitting - Marginal

Theorem [Valiquette et al. (2025)]

Let $p \in (0,1)$, $\alpha > 0$, $\boldsymbol{a} = (a_1, \ldots, a_K)$ and $\boldsymbol{b} = (b_1, \ldots, b_K)$ two positive vectors with $K \geq 1$. If X is distributed as in (2), then its p.m.f. is given by

$$\left[\prod_{k=1}^{K} \frac{(\alpha_k)_n}{(\alpha_k + \beta_k)_n}\right] \frac{(\alpha)_n}{n!} \left(\frac{p}{1-p}\right)^n \sum_{m=0}^{\infty} \frac{(\alpha+n)_m}{m!} p^m (1-p)^{\alpha+n} {}_{K+1}F_K \left[\begin{matrix} -m, \ \boldsymbol{a} + n\mathbf{1} \\ \boldsymbol{a} + \boldsymbol{b} + n\mathbf{1} \end{matrix}; 1\right],$$

where $\mathbf{1}$ is the unit vector.

Tree Pólya Splitting - Marginal

Theorem [Valiquette et al. (2025)]

In particular, if $p \in (0, 1/2)$, its p.m.f. is given by

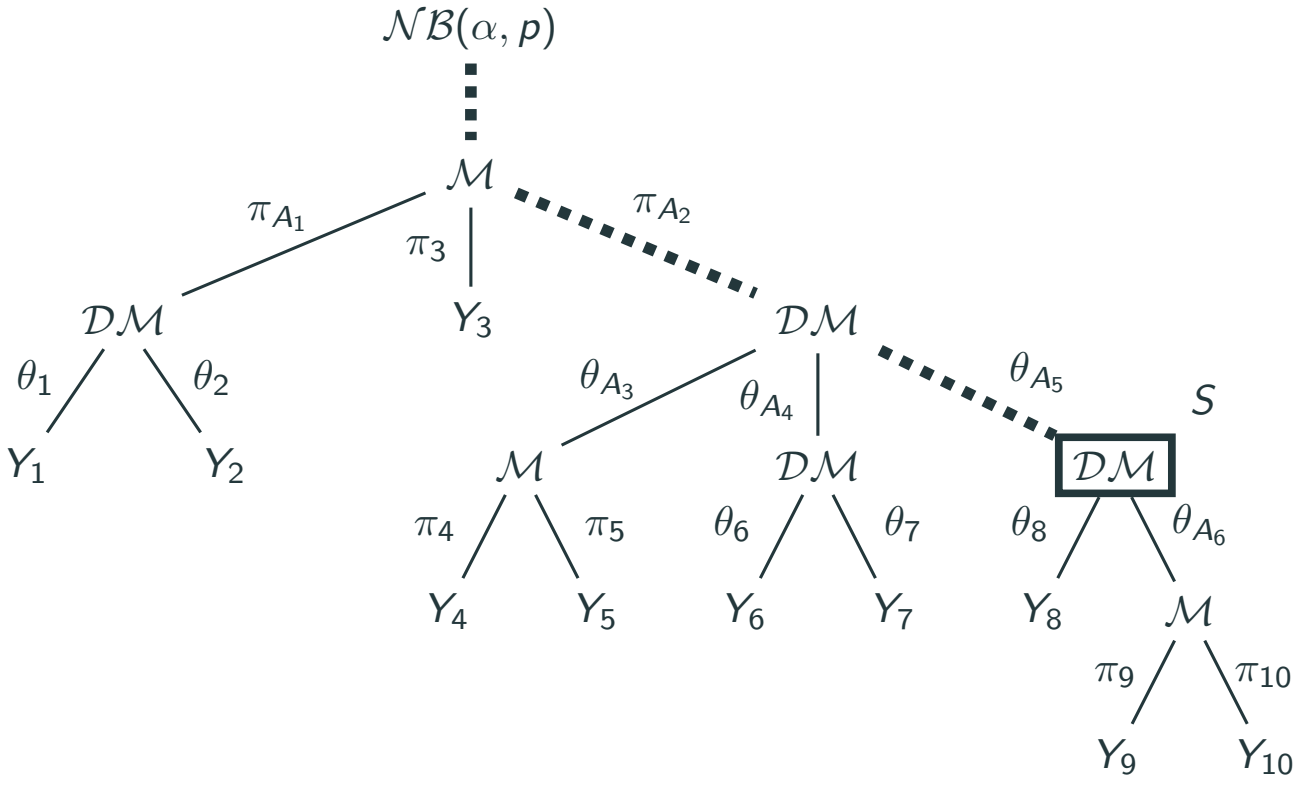
$$\left[\prod_{k=1}^{K} \frac{(\alpha_k)_n}{(\alpha_k + \beta_k)_n}\right] \frac{(\alpha)_n}{n!} \left(\frac{p}{1-p}\right)^n {}_{K+1}F_K\left[\begin{matrix} \alpha + n,\ \boldsymbol{a} + n\mathbf{1} \\ \boldsymbol{a} + \boldsymbol{b} + n\mathbf{1} \end{matrix}; \frac{p}{p-1}\right].$$

Tree Pólya Splitting - Covariance & correlation

Let $A \in \mathfrak{I} \cup \mathfrak{L}$ and $A_k \in \mathrm{Path}_A$, we define the following constants.

$$\gamma_A = \prod_{k=1}^{K} \frac{\theta_{A_{k-1}}}{|\boldsymbol{\theta}_{A_k}|} \text{ and } \delta_A = \prod_{k=1}^{K} \frac{\theta_{A_{k-1}} + c_{A_k}}{|\boldsymbol{\theta}_{A_k}| + c_{A_k}}.$$

Tree Pólya Splitting - Covariance & correlation



$$\gamma_S = \frac{\theta_{A_5}}{\theta_{A_3}+\theta_{A_4}+\theta_{A_5}} \cdot \pi_{A_2} \text{ and } \delta_S = \frac{\theta_{A_5}+1}{\theta_{A_3}+\theta_{A_4}+\theta_{A_5}+1} \cdot \pi_{A_2}$$

Tree Pólya Splitting - Covariance & correlation

Theorem [Valiquette et al. (2025)]

Let $\mathbf{Y} \sim \mathcal{TP}_{\Delta_n}(\mathfrak{T}; \boldsymbol{\theta}, \mathbf{c}) \underset{n}{\wedge} \mathcal{L}(\boldsymbol{\psi})$ and marginals Y_i, Y_j $(i \neq j)$. Then, there is a common ancestor node $S \in \mathfrak{I}$ such that

$$\mathrm{Cov}(Y_i, Y_j) = \gamma_i \gamma_j \left[\left(\frac{|\boldsymbol{\theta}_S|}{|\boldsymbol{\theta}_S| + c_S} \right) \frac{\delta_S}{\gamma_S} \mu_2 - \mu_1^2 \right].$$

Note: The sign only depends on S, which is dependent on the choice of Y_i and Y_j.

Tree Pólya Splitting - Covariance & correlation

Theorem [Valiquette et al. (2025)]

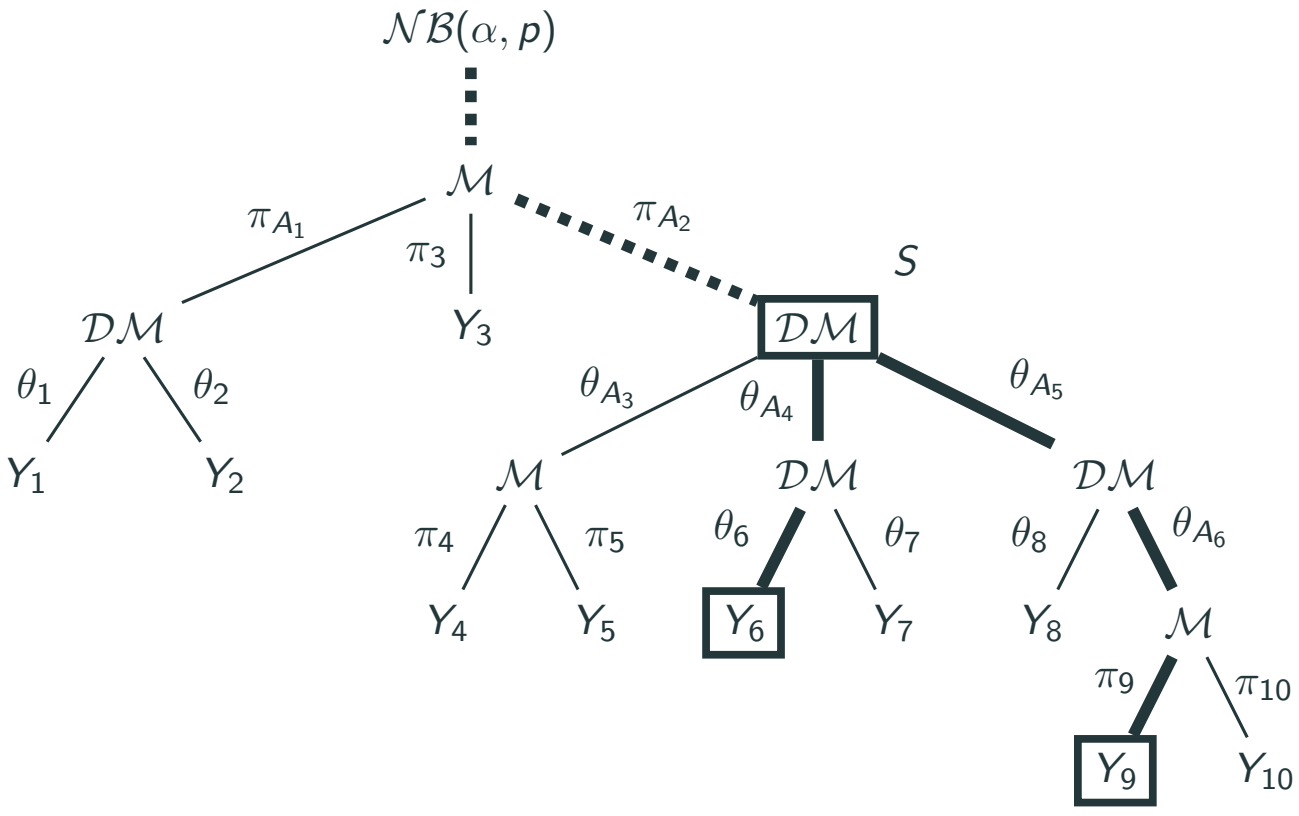
Let $\mathbf{Y} \sim \mathcal{TP}_{\Delta_n}(\mathfrak{T}; \boldsymbol{\theta}, \mathbf{c}) \underset{n}{\wedge} \mathcal{L}(\boldsymbol{\psi})$ and marginals Y_i, Y_j $(i \neq j)$. Then, there is a common ancestor node $S \in \mathfrak{I}$ such that

$$\mathrm{Corr}(Y_i, Y_j) = \Lambda_i \Lambda_j \left[\left(\frac{|\boldsymbol{\theta}_S|}{|\boldsymbol{\theta}_S| + c_S} \right) \frac{\delta_S}{\gamma_S} \mu_2 - \mu_1^2 \right],$$

where

$$\Lambda_k = \sqrt{\frac{\gamma_\ell}{\delta_\ell \mu_2 + \mu_1 (1 - \gamma_\ell \mu_1)}};\ k = i,j.$$

Tree Pólya Splitting - Covariance & correlation



$$\mathrm{Cov}(Y_6, Y_9) = \alpha \left(\frac{p}{1-p}\right)^2 \left(\frac{|\boldsymbol{\theta}_S| - \alpha}{|\boldsymbol{\theta}_S| + 1}\right) \gamma_6 \gamma_9$$

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In this example, the sum at S is such that

$$|\mathbf{Y}_S| \sim \mathcal{NB}\left(\alpha, \frac{p\pi_{A_2}}{1-p+p\pi_{A_2}}\right),$$

and (Y_6, Y_9) are independent if and only if $\alpha = \theta_{A_3} + \theta_{A_4} + \theta_{A_5}$ (Peyhardi et al. [2021]).

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In fact, the sets $\{Y_4, Y_5\}$, $\{Y_6, Y_7\}$, and $\{Y_8, Y_9, Y_{10}\}$ are mutually independent if and only if

$$\alpha = \theta_{A_3} + \theta_{A_4} + \theta_{A_5},$$

since all their common ancestor is the node S.

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Let us fix the following parameters:

$$\boldsymbol{\alpha = 10}$$

$$p = 0.95$$

$$\pi_{A_1} = \pi_9 = 0.3$$

$$\pi_3 = 0.1$$

$$\theta_1 = \theta_2 = 1.5$$

$$\pi_{A_2} = 0.6$$

$$\boldsymbol{\theta_{A_3} = 3}$$

$$\boldsymbol{\theta_{A_4} = \theta_{A_5} = 3.5}$$

$$\pi_4 = \pi_5 = 0.5$$

$$\theta_6 = 0.8$$

$$\theta_7 = \theta_8 = 1$$

$$\theta_{A_6} = 2.5$$

$$\pi_{10} = 0.7$$

Tree Pólya Splitting - Covariance & correlation

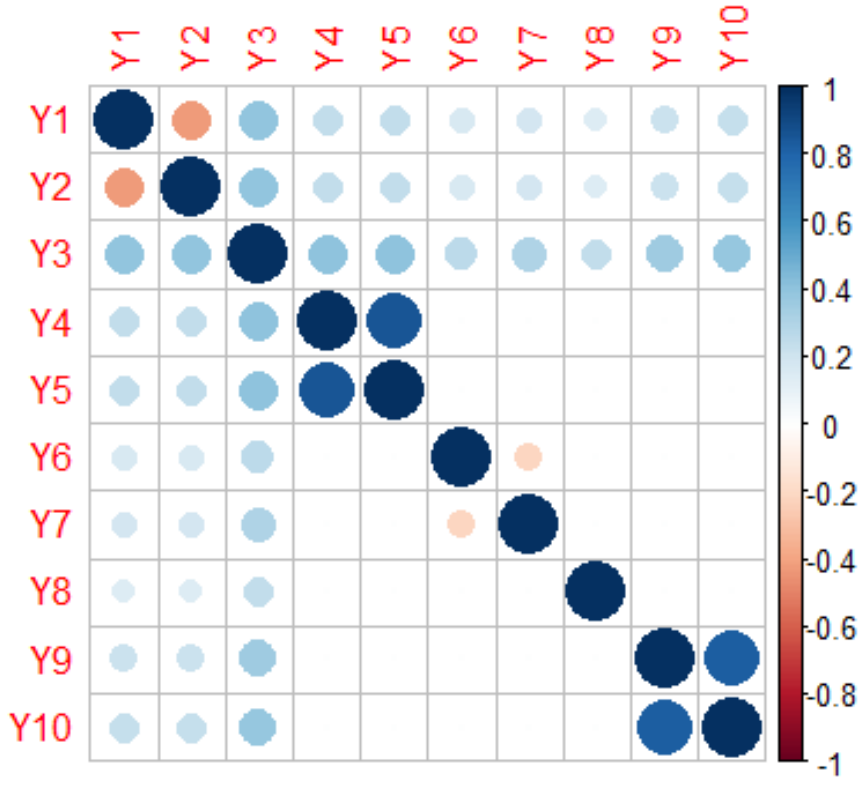

Figure 3: Correlation plot of the example

Application to Trichoptera dataset

Application to Trichoptera dataset

Tree Pólya Splitting is adjusted to the **Trichoptera dataset** provided by Usseglio-Polatera & Auda [1987]. It consists of $\mathbf{J = 17}$ **species'** abundances on $\mathbf{n = 49}$ **sites** between 1959 and 1960.

Application to Trichoptera dataset

Che	Hyc	Hym	Hys	Psy	Aga	Glo	Ath	Cea	Ced	Set
0	0	5	0	17	0	0	0	0	2	0
0	0	3	0	8	0	0	0	0	0	0
0	0	1	0	32	0	0	0	0	0	0
0	0	3	0	176	4	0	0	0	1	0
0	0	4	0	69	2	0	0	0	0	0
0	0	2	0	14	1	0	0	0	0	0

Table 1: Trichoptera dataset

Application to Trichoptera dataset

Using the AIC criterion, we built $\mathfrak{T}$ using the dataset and compared our model to other **Pólya Splitting** distributions and to the **multivariate Poisson-lognormal** (Chiquet et al. [2021]).

Application to Trichoptera dataset

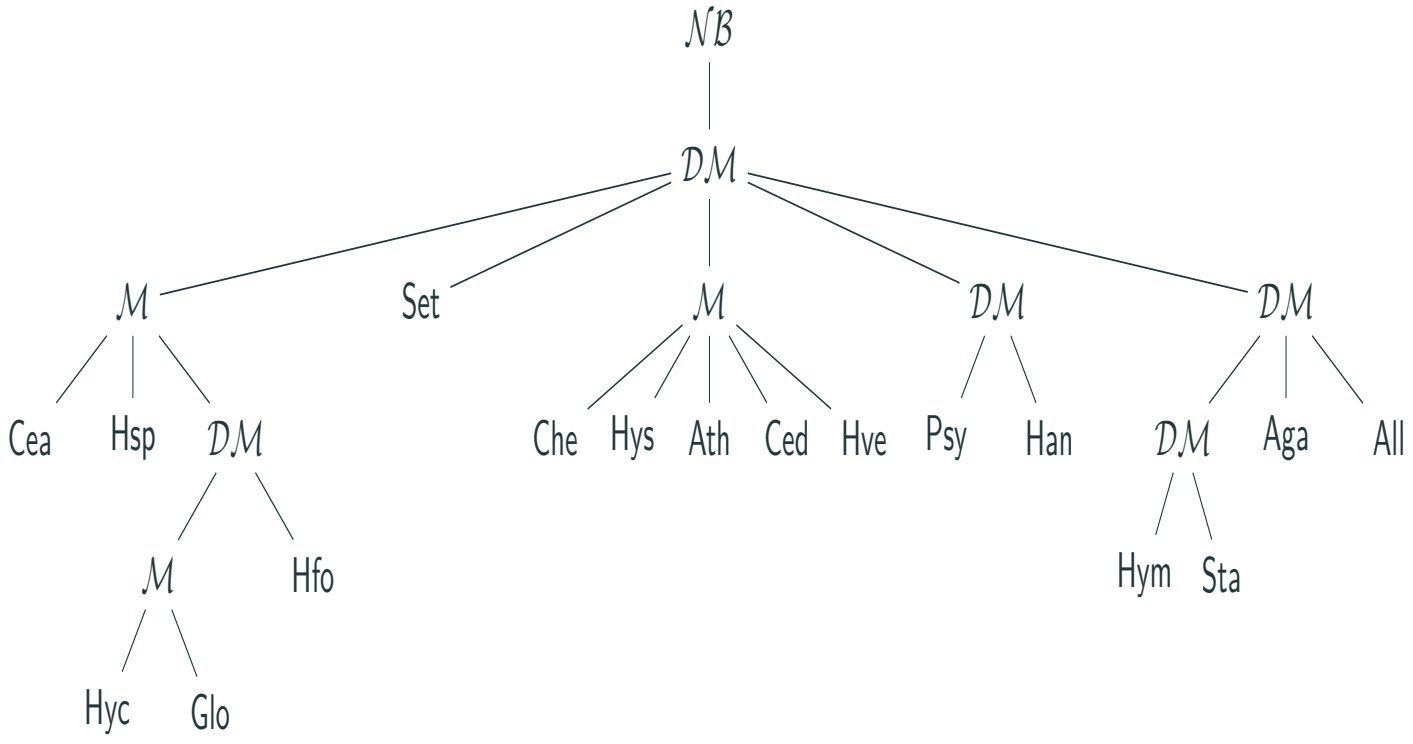

Figure 4: Tree Pólya Splitting fitted to the Trichoptera data.

Application to Trichoptera dataset

Models	Nb. Parameters	AIC
Tree Pólya Splitting	23	2380.77
Generalized Dirichlet-multinomial Splitting	34	2460.70
Dirichlet-multinomial Splitting	19	2494.87
Multivariate Poisson-lognormal	170	2599.63
Multinomial Splitting	18	6362.20

Table 2: Fitted models to the Trichoptera data.

Application to Trichoptera dataset

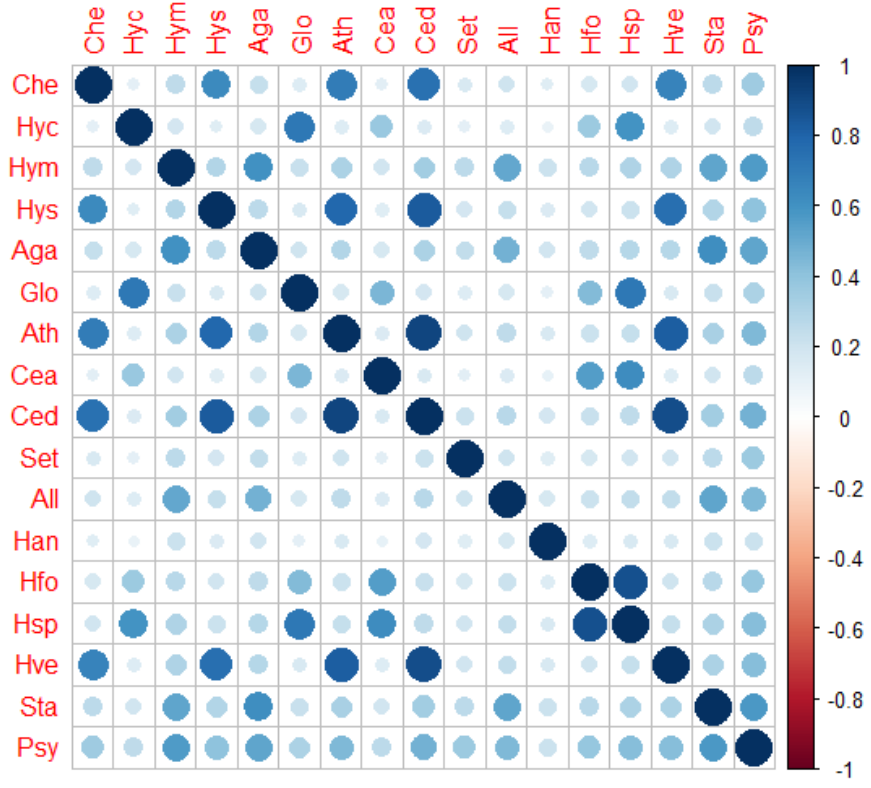

Figure 5: Correlation plot of the Tree Pólya Splitting fitted to the Trichoptera data.

Conclusion and perspectives

Conclusions and perspectives

Summary

- Tree Pólya Splitting is a flexible and simple model for multivariate count data.
- Its marginals, moments, correlations, and independences can be easily evaluated.

Perspectives

- Incorporating covariates at each internal node.
- Using the tree structure for multivariate zero inflation (Moudjeu et al. [2025]).
- Developing a probabilistic graphical model for such distributions.

Thank you for your attention!

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Article: Tree Pólya Splitting distributions for multivariate count data (arXiv:2404.19528)

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